

SECOND-ORDER SUBGRADIENTS OF CONVEX INTEGRAL FUNCTIONALS

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ABSTRACT. The purpose of this work is twofold: on the one hand, we study the second-order behaviour of a nonsmooth convex function F defined over a reflexive Banach space X . We establish several equivalent characterizations of the set $\partial^2 F(\bar{x}, \bar{y})$, known as the second-order subdifferential of F at \bar{x} relative to $\bar{y} \in \partial F(\bar{x})$. On the other hand, we examine the case in which $F = I_f$ is the functional integral associated to a normal convex integrand f . We extend a result of Chi Ngoc Do from the space $X = L^p_{\mathbb{R}^d}$ ($1 < p < +\infty$) to a possible nonreflexive Banach space $X = L^p_E$ ($1 \leq p < +\infty$). We also establish a formula for computing the second-order subdifferential $\partial^2 I_f(\bar{x}, \bar{y})$.

1. INTRODUCTION

There is a very rich literature devoted to the study of convex integral functionals of the form

$$x \in X \mapsto I_f(x) := \int_S f(s, x(s)) \mu(ds).$$

Such functionals arise very often in the formulation of variational problems, and have received a great deal of attention since the late sixties. Rockafellar's formula [Ro2, Ro3] for computing the Legendre-Fenchel conjugate $(I_f)^*$ and the subdifferential mapping ∂I_f are now well known and widely used. Bustos's characterization [Bu] of the ε -subdifferential mapping $\partial_\varepsilon I_f$ has also found several interesting applications. The problem of the convergence of a sequence $\{I_{f_n}\}_{n \in \mathbb{N}}$ of convex integral functionals has been explored by authors like Joly and de Thelin [JT] and Salvadori [Sa].

This paper is concerned with the second-order analysis of I_f . The case of a smooth integrand f has been studied by Borwein and Noll [BN], and Noll [No1, No2], among other authors. However, there is still a lot of room left for the exploration of the nonsmooth case.

Information on the second-order behaviour of I_f around a given point \bar{x} can be obtained by taking a limit of the form

$$(1.1) \quad I_f''(\bar{x}, \bar{y}; \cdot) = \text{"} \lim_{t \rightarrow 0^+} \text{" } \delta_t^2 I_f(\bar{x}, \bar{y}; \cdot),$$

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where

$$\delta_t^2 I_f(\bar{x}, \bar{y}; h) := \frac{2}{t} \left[\frac{I_f(\bar{x} + th) - I_f(\bar{x})}{t} - \langle \bar{y}, h \rangle \right]$$

represents a second-order differential quotient, \bar{y} is a particular subgradient in $\partial I_f(\bar{x})$, and $\langle \bar{y}, \cdot \rangle$ is the corresponding continuous linear form on X . The expression (1.1) represents a (generalized) second derivative of I_f at \bar{x} relative to \bar{y} . It depends of course on the convergence notion used in the definition of the limit.

Recently, Chi Ngoc Do established in his thesis [Do1] a formula for computing I_f'' in terms of the (generalized) second derivative of the function $f(s, \cdot)$. Do's result can be considered as the starting point of our work.

The organization of the paper is as follows. In Section 2 we establish some results concerning the second derivative $F''(\bar{x}, \bar{y}; \cdot)$ of a general convex function F defined over a reflexive Banach space X . In fact, we focus our attention toward a “dual” concept associated to $F''(\bar{x}, \bar{y}; \cdot)$; namely, the second-order subdifferential $\partial^2 F(\bar{x}, \bar{y})$. Section 3 concerns the case in which F is a convex integral functional. We extend the result of Do from the space $X = L_{\mathbb{R}^d}^p$ ($1 < p < +\infty$) to a possible nonreflexive Banach space $X = L_E^p$ ($1 \leq p < +\infty$). We also establish a formula for computing the second-order subdifferential $\partial^2 I_f(\bar{x}, \bar{y})$. Section 4 is left to the discussion of two important applications.

2. SECOND-ORDER SUBGRADIENTS OF CONVEX FUNCTIONS

Throughout this section X denotes a reflexive Banach space with topological dual X^* , and $\langle \cdot, \cdot \rangle$ stands for the canonical pairing between X and X^* , i.e.,

$$\langle y, x \rangle := y(x) \quad \text{for all } y \in X^*, x \in X.$$

We are concerned with the study of the second-order behaviour of a proper convex function $F: X \rightarrow \mathbb{R} \cup \{+\infty\}$ around a given point \bar{x} in its effective domain

$$\text{dom } F := \{x \in X : F(x) < +\infty\}.$$

Recall that the first-order behaviour of F around \bar{x} is reflected by the set

$$(2.1) \quad \partial F(\bar{x}) := \{y \in X^* : F(x) \geq F(\bar{x}) + \langle y, x - \bar{x} \rangle \text{ for all } x \in X\},$$

known as the *subdifferential* of F at \bar{x} . The elements y in $\partial F(\bar{x})$ are called *subgradients* of F at \bar{x} . In order to calculate the subdifferential of F , it helps sometimes to work with the Legendre-Fenchel conjugate of F , which is the function $F^*: X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F^*(y) := \sup_{x \in X} \{\langle y, x \rangle - F(x)\} \quad \text{for all } y \in X^*.$$

The following standard result can be found, for instance, in the books by Barbu and Precupanu [BP, Chapter 2.2], and Rockafellar [Ro1, Theorem 23.5].

Proposition 2.1. *Let $F: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then, the following three conditions are equivalent:*

$$(2.2) \quad y \in \partial F(x);$$

$$(2.3) \quad F^*(y) + F(x) - \langle y, x \rangle \leq 0;$$

$$(2.4) \quad F^*(y) + F(x) - \langle y, x \rangle = 0.$$

If in addition F is lower-semicontinuous, then all these conditions are equivalent to the following one:

$$(2.5) \quad x \in \partial F^*(y).$$

From now on we assume that $F: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex lower-semicontinuous function. In short, F belongs to the class $\Gamma_0(X)$.

Second-order information on the behaviour of F around the point \bar{x} is in general more difficult to obtain. First, we have to pick up a particular subgradient \bar{y} in $\partial F(\bar{x})$, and then we have to examine what happens with the second-order differential quotient

$$(2.6) \quad \delta_t^2 F(\bar{x}, \bar{y}, h) := \frac{2}{t} \left[\frac{F(\bar{x} + th) - F(\bar{x})}{t} - \langle \bar{y}, h \rangle \right]$$

as the increment t tends to 0^+ . In this paper, the convergence of the family of functions $\{\delta_t^2 F(\bar{x}, \bar{y}; \cdot)\}_{t>0}$ toward a limit function $F''(\bar{x}, \bar{y}; \cdot)$ will be understood in the sense of Mosco. As pointed out by Do [Do1, Do2], this type of convergence yields a second-order derivative concept which has good duality and variational properties. To make this paper self-contained, we start by recalling:

Definition 2.1. Let $\overline{\mathbb{R}}$ denote the set of extended real numbers, i.e., $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. A sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of functions $\varphi_n: X \rightarrow \overline{\mathbb{R}}$ is said to converge in the sense of Mosco toward the function $\varphi: X \rightarrow \overline{\mathbb{R}}$ if for every $h \in X$, one has

$$(2.7) \quad \forall \{h_n\} \xrightarrow{w} h, \quad \varphi(h) \leq \liminf \varphi_n(h_n),$$

$$(2.8) \quad \exists \{h_n\} \xrightarrow{s} h, \quad \varphi(h) \geq \limsup \varphi_n(h_n).$$

A family $\{\varphi_t\}_{t>0}$ of functions $\varphi_t: X \rightarrow \overline{\mathbb{R}}$ is said to converge in the sense of Mosco toward $\varphi: X \rightarrow \overline{\mathbb{R}}$ if for every $\{t_n\} \rightarrow 0^+$, the sequence $\{\varphi_{t_n}\}_{n \in \mathbb{N}}$ Mosco converges toward φ . In such a case one writes $\varphi = \text{M-lim}_{t \rightarrow 0^+} \varphi_t$, and one says that φ is the Mosco limit of the family $\{\varphi_t\}_{t>0}$.

The symbols \xrightarrow{w} and \xrightarrow{s} have been used in Definition 2.1 to denote, respectively, the convergence with respect to the weak and strong topologies in X . If X is finite dimensional, then both topologies coincide and the Mosco convergence is nothing but the so-called epigraphical convergence. General properties of Mosco limits can be found, for instance, in the original papers by Mosco [Mo1, Mo2], or in the book by Attouch [At]. All that the reader needs to know about Mosco limits of second-order differential quotients can be summarized in the following proposition.

Proposition 2.2. Let $F \in \Gamma_0(X)$ be finite at $\bar{x} \in X$, and let $\bar{y} \in \partial F(\bar{x})$. Suppose that the Mosco limit

$$(2.9) \quad F''(\bar{x}, \bar{y}; \cdot) := \text{M-lim}_{t \rightarrow 0^+} \delta_t^2 F(\bar{x}, \bar{y}; \cdot)$$

exists. Then, there is a unique nonempty closed convex set C in X^* such that

$$(2.10) \quad F''(\bar{x}, \bar{y}; h) = \left[\sup_{z \in C} \langle z, h \rangle \right]^2 \quad \text{for all } h \in X.$$

Moreover, the set C contains the origin $0 \in X^*$.

Proof. Just observe that $F''(\bar{x}, \bar{y}; \cdot)$ is a nonnegative convex lower-semicontinuous function satisfying

$$F''(\bar{x}, \bar{y}; 0) = 0 \text{ and } F''(\bar{x}, \bar{y}; \alpha h) = \alpha^2 F''(\bar{x}, \bar{y}; h) \text{ for all } \alpha > 0 \text{ and } h \in X.$$

By taking the square root of $F''(\bar{x}, \bar{y}; \cdot)$ one gets a proper lower-semicontinuous sublinear function. To complete the proof it suffices to apply Hörmander's theorem [Ho]. \square

If one takes into account Proposition 2.2, it is clear that the above set C plays an important role in the second-order analysis of the function F . An explicit characterization of this set is given in the next definition.

Definition 2.2. Let $F \in \Gamma_0(X)$ be finite at $\bar{x} \in X$, and let $\bar{y} \in \partial F(\bar{x})$. If the Mosco limit $F''(\bar{x}, \bar{y}; \cdot)$ exists, then the set

$$(2.11) \quad \partial^2 F(\bar{x}, \bar{y}) = \{z \in X^* : \langle z, h \rangle \leq [F''(\bar{x}, \bar{y}; h)]^{1/2} \text{ for all } h \in X\}$$

is referred to as the second-order subdifferential of F at \bar{x} relative to \bar{y} . Each element z in $\partial^2 F(\bar{x}, \bar{y})$ is called a second-order subgradient of F at \bar{x} relative to \bar{y} .

The expression (2.11) is not entirely new. A variant of the set (2.11) can be obtained by using pointwise convergence ([Hi], [Se1], [HS1], [HS2], [HS3]). In a finite dimensional setting, the Mosco limit (2.9) coincides with the epigraphical limit

$$F''(\bar{x}, \bar{y}; \cdot) = \operatorname{epi} \lim_{t \rightarrow 0^+} \delta_t^2 F(\bar{x}, \bar{y}; \cdot)$$

introduced by Rockafellar [Ro4, Ro5, Ro6]. The associated set (2.11) has been studied in [Se2], [Se3], [Se4], and [Pe].

In the same way as Seeger [Se4, Theorem 4.1] did in a finite dimensional context, we characterize now the second-order subdifferential $\partial^2 F(\bar{x}, \bar{y})$ in terms of the Mosco limit of a family of sets $\{\partial_{[t]}^2 F(\bar{x}, \bar{y})\}_{t>0}$ in X^* . Each set in this family is defined by

$$(2.12) \quad \partial_{[t]}^2 F(\bar{x}, \bar{y}) := \frac{\partial_{[t]} F(\bar{x}) - \bar{y}}{t},$$

and is regarded as an *approximate second-order subdifferential* of F at \bar{x} relative to \bar{y} . Here

$$(2.13) \quad \partial_{[t]} F(\bar{x}) := \left\{ y \in X^* : F^*(y) + F(\bar{x}) - \langle y, \bar{x} \rangle \leq \frac{t^2}{2} \right\}$$

corresponds to an approximate (first-order) subdifferential of F at \bar{x} . Observe that the set (2.13) is obtained by setting $\varepsilon = t^2/2$ in the definition of the classical ε -subdifferential

$$\partial_\varepsilon F(\bar{x}) := \{y \in X^* : F(x) \geq F(\bar{x}) + \langle y, x - \bar{x} \rangle - \varepsilon \text{ for all } x \in X\}.$$

Before stating the main result of this section, we recall the concept of Mosco convergence of sets in X^* , and prove a technical lemma concerning the Mosco convergence of sublevel sets.

Definition 2.3. A sequence $\{S_n\}_{n \in \mathbb{N}}$ of subsets in X^* is said to be Mosco convergent if the weak upper limit

$$w\text{-}\limsup S_n := \{z \in X^* : \exists \{n_k\}_{k \in \mathbb{N}}, \exists \{z_k\} \xrightarrow{w} z \text{ with } z_k \in S_{n_k} \text{ for all } k \in \mathbb{N}\}$$

coincides with the strong lower limit

$$s\text{-}\liminf S_n := \{z \in X^* : \exists \{z_n\} \xrightarrow{s} z \text{ with } z_n \in S_n \text{ for } n \text{ sufficiently large}\}.$$

The common value is then denoted by $M\text{-}\lim S_n$. A family $\{S_t\}_{t>0}$ of subsets in X^* is said to be Mosco convergent if for every $\{t_n\} \rightarrow 0^+$, the sequence $\{S_{t_n}\}_{n \in \mathbb{N}}$ is Mosco convergent. In such a case the Mosco limit $M\text{-}\lim S_{t_n}$ does not depend on the choice of $\{t_n\} \rightarrow 0^+$, and is denoted simply by $M\text{-}\lim_{t \rightarrow 0^+} S_t$.

The first part of the next lemma seems to be a new result. It was suggested to us by Michel Volle (Avignon), to whom we express our acknowledgement. The second part is taken from Mosco [Mo2, Lemma 3.1]. The notation

$$\{g \leq \alpha\} := \{z \in X^* : g(z) \leq \alpha\}$$

refers to the sublevel of g associated to the height $\alpha \in \mathbb{R}$. Of course, $\{g < \alpha\}$ is the corresponding strict sublevel set. Several authors have studied the convergence of functions in terms of the convergence of their sublevel sets. Variants of Lemma 2.1 can be found, for instance, in Beer and Luchetti [BL, Theorem 5.1], Beer, Rockafellar, and Wets [BRW, Theorem 3.1], and Soueiyatt [So, Corollary 2.6].

Lemma 2.1. *Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of extended-real-valued functions defined on X^* , and let g be another such function. Then, the following statements are equivalent:*

$$(2.14) \quad \{g_n\}_{n \in \mathbb{N}} \text{ Mosco converges toward } g;$$

$$(2.15) \quad \text{for all } \alpha \in \mathbb{R}, \text{ one has} \\ w\text{-}\limsup\{g_n \leq \alpha\} \subset \{g \leq \alpha\} \quad \text{and} \quad s\text{-}\liminf\{g_n \leq \alpha\} \supset \{g < \alpha\}.$$

Moreover, if $\{g_n\}_{n \in \mathbb{N}}$ Mosco converges toward a convex function g , and if α is strictly greater than $\inf_{X^*} g$, then

$$M\text{-}\lim\{g_n \leq \alpha\} = \{g \leq \alpha\}.$$

Proof. The lemma obtains from more abstract convergence results stated without proof in [Vo]. For the sake of completeness we give here a detailed proof of the first part of the lemma. Let us start with the implication (2.14) \Rightarrow (2.15). Suppose that $\{g_n\}$ Mosco converges to g , and let $\alpha \in \mathbb{R}$. If $z \in w\text{-}\limsup\{g_n \leq \alpha\}$, then there exist a subsequence $\{n_k\}_{k \in \mathbb{N}}$ and a sequence $\{z_k\}_{k \in \mathbb{N}} \xrightarrow{w} z$ such that $z_k \in \{g_{n_k} \leq \alpha\}$. Clearly, one has

$$(2.16) \quad \liminf_k g_{n_k}(z_k) \leq \alpha.$$

Moreover, the Mosco convergence of $\{g_n\}$ toward g allows us to write

$$(2.17) \quad g(z) \leq \liminf_k g_{n_k}(z_k).$$

By combining (2.16) and (2.17), one gets $g(z) \leq \alpha$. In this way, we have proven that

$$w\text{-}\limsup\{g_n \leq \alpha\} \subset \{g \leq \alpha\}.$$

Now take z in $\{g < \alpha\}$. Since $\{g_n\}$ Mosco converges to g , there exists a sequence $\{z_n\} \xrightarrow{s} z$ such that $\lim_n g_n(z) \leq g(z)$. Since $g(z) < \alpha$, it is clear that $g_n(z_n) \leq \alpha$ for all n sufficiently large. In this way, we have proven that

$$\{g < \alpha\} \subset s\text{-}\liminf\{g_n \leq \alpha\}.$$

Consider now the reverse implication (2.15) \Rightarrow (2.14). Let $z \in X^*$ be arbitrary. First, we prove

$$(2.18) \quad \forall \{z_n\} \xrightarrow{w} z, \quad g(z) \leq \liminf g_n(z_n).$$

Take $\{z_n\} \xrightarrow{w} z$ and suppose that the above inequality does not hold. This means that

$$(2.19) \quad g(z) > \beta > \liminf g_n(z_n)$$

for some $\beta \in \mathbb{R}$. Now take a subsequence $v_k := z_{n_k}$ such that

$$\lim_k g_{n_k}(v_k) = \liminf g_n(z_n).$$

Clearly, $v_k \xrightarrow{w} z$ and $v_k \in \{g_{n_k} \leq \beta\}$ for all k sufficiently large. In fact, we can suppose that the later condition holds for all $k \in \mathbb{N}$. Hence, z belongs to $w\text{-}\limsup\{g_n \leq \beta\}$. By assumption, the later set is contained in $\{g \leq \beta\}$. One gets in this way $g(z) \leq \beta$, which contradicts (2.19). Now, we prove that

$$(2.20) \quad \exists \{z_n\} \xrightarrow{s} z \text{ such that } g(z) \geq \limsup g_n(z_n).$$

If $g(z) = +\infty$, then there is nothing to prove. Otherwise, we set, for all $k \in \mathbb{N} \setminus \{0\}$,

$$\alpha_k := \begin{cases} g(z) + k^{-1} & \text{if } g(z) \in \mathbb{R}, \\ -k & \text{if } g(z) = -\infty. \end{cases}$$

Hence, z belongs to $\{g < \alpha_k\}$. By assumption, the later set is contained in $s\text{-}\liminf_n \{g_n \leq \alpha_k\}$. Consequently, for each $k \in \mathbb{N} \setminus \{0\}$, there exists a sequence $\{z_n^k\}_{n \in \mathbb{N}} \xrightarrow{s} z$ such that $z_n^k \in \{g_n \leq \alpha_k\}$ for all n sufficiently large. According to a standard diagonalization procedure, there exists a strictly increasing function $\varphi: \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\}$ such that $z_n = z_n^{\varphi(n)} \xrightarrow{s} z$. Since $g_n(z_n) \leq \alpha_{\varphi(n)}$ and $\lim_n \alpha_{\varphi(n)} = g(z)$, it follows that $\limsup g_n(z_n) \leq g(z)$. Condition (2.20) is proven in this way. The proof of the implication (2.15) \Rightarrow (2.14) is now complete. \square

Now we are ready to extend Seeger's result [Se4, Theorem 4.1] to an infinite dimensional context. In fact, the proof of (2.22) \Rightarrow (2.21) is entirely new, since this implication does not appear in [Se4].

Theorem 2.1. *Let $F \in \Gamma_0(X)$ be finite at $\bar{x} \in X$, and let $\bar{y} \in \partial F(\bar{x})$. Then, the following conditions are equivalent:*

$$(2.21) \quad \text{the Mosco limit } F''(\bar{x}, \bar{y}; \cdot) \text{ exists;}$$

$$(2.22) \quad \text{the family } \{\partial_{[t]}^2 F(\bar{x}, \bar{y})\}_{t>0} \text{ is Mosco convergent.}$$

Moreover, under the above equivalent conditions, one can write

$$(2.23) \quad \partial^2 F(\bar{x}, \bar{y}) = \text{M-} \lim_{t \rightarrow 0^+} \partial_{[t]}^2 F(\bar{x}, \bar{y}).$$

Proof. As observed by the second author in [Se1, Proposition B.4.7] and [Se4, Proposition 2.2], the approximate second-order subdifferential $\partial_{[t]}^2 F(\bar{x}, \bar{y})$ can be expressed as a sublevel set involving the second-order differential quotient

$$\delta_t^2 F^*(\bar{y}, \bar{x}; v) := \frac{2}{t} \left[\frac{F^*(\bar{y} + tv) - F^*(\bar{y})}{t} - \langle v, \bar{x} \rangle \right].$$

Indeed, one can write

$$(2.24) \quad \partial_{[t]}^2 F(\bar{x}, \bar{y}) = \{v \in X^*: \delta_t^2 F^*(\bar{y}, \bar{x}; v) \leq 1\} \quad \text{for all } t > 0.$$

Thus, we are concerned with the Mosco convergence of a family of sublevel sets. Lemma 2.1 will be, in this respect, of great help. Assume first that the Mosco limit

$$F''(\bar{x}, \bar{y}; \cdot) = \text{M-} \lim_{t \rightarrow 0^+} \delta_t^2 F(\bar{x}, \bar{y}; \cdot)$$

exists. As noticed by Do [Do2, Theorem 2.5], from the equality

$$\frac{1}{2} \delta_t^2 F^*(\bar{y}, \bar{x}; v) = [\frac{1}{2} \delta_t^2 F(\bar{x}, \bar{y}; \cdot)]^*(v)$$

and Wijsman-Mosco's theorem on the continuity of the Legendre-Fenchel transform (cf. [Mo2, Theorem 1]), one deduces the existence of the Mosco limit

$$(F^*)''(\bar{y}, \bar{x}; \cdot) = \text{M-} \lim_{t \rightarrow 0^+} \delta_t^2 F^*(\bar{y}, \bar{x}; \cdot).$$

Since

$$1 > \inf_{v \in X^*} (F^*)''(\bar{y}, \bar{x}; v) = 0,$$

the second part of Lemma 2.1 yields the Mosco convergence of the family $\partial_{[t]}^2 F(\bar{x}, \bar{y})$ toward the set

$$C := \{v \in X^* : (F^*)''(\bar{y}, \bar{x}; v) \leq 1\}.$$

In this way we have proven that (2.21) implies (2.22). Assume now that (2.22) holds. To prove (2.21) we will examine the sublevel set

$$A_t(\alpha) := \{v \in X^* : \delta_t^2 F^*(\bar{y}, \bar{x}; v) \leq \alpha\}.$$

Since $\{\delta_t^2 F^*(\bar{y}, \bar{x}; \cdot)\}_{t>0}$ is a family of nonnegative functions, it suffices to consider the case $\alpha \geq 0$. For the height $\alpha = 1$, we know that $A_t(1) = \partial_{[t]}^2 F(\bar{x}, \bar{y})$ Mosco converges. The Mosco limit

$$D := \text{M-} \lim_{t \rightarrow 0^+} \partial_{[t]}^2 F(\bar{x}, \bar{y})$$

is necessarily a closed convex set in X^* which contains the origin $0 \in X^*$. Thus, D can be expressed as the sublevel set

$$D = \{g \leq 1\}$$

where

$$g(v) := \left[\sup_{u \in D^0} \langle u, v \rangle \right]^2 \quad \text{for all } v \in X^*,$$

and

$$D^0 := \{u \in X : \langle v, u \rangle \leq 1 \text{ for all } v \in D\}$$

denotes the polar set of D . For a height $\alpha > 0$, one can write

$$\begin{aligned} A_t(\alpha) &= \left\{ v \in X^* : \delta_{\sqrt{\alpha}t}^2 F^* \left(\bar{y}, \bar{x}; \frac{1}{\sqrt{\alpha}} v \right) \leq 1 \right\} \\ &= \sqrt{\alpha} \{ v \in X^* : \delta_{\sqrt{\alpha}t}^2 F^*(\bar{y}, \bar{x}; v) \leq 1 \} \\ &= \sqrt{\alpha} A_{\sqrt{\alpha}t}(1). \end{aligned}$$

Thus, the Mosco limit

$$\text{M-} \lim_{t \rightarrow 0^+} A_t(\alpha) = \sqrt{\alpha} \text{M-} \lim_{\tau \rightarrow 0^+} A_\tau(1) = \sqrt{\alpha} \{g \leq 1\} = \{g \leq \alpha\}$$

exists as well. Finally, consider the height $\alpha = 0$. Since $\{g < 0\}$ is empty, the inclusion

$$s\text{-}\liminf_{t \rightarrow 0^+} A_t(0) \supset \{g < 0\}$$

trivially holds. To cover all the possible cases in statement (2.15) of Lemma 2.1, it remains only to check the inclusion

$$w\text{-}\limsup_{t \rightarrow 0^+} A_t(0) \subset \{g \leq 0\}.$$

Observe that

$$\begin{aligned} A_t(0) &= \{v \in X^* : \delta_t^2 F^*(\bar{y}, \bar{x}; v) \leq 0\} \\ &= \{v \in X^* : F^*(\bar{y} + tv) - F^*(\bar{y}) - t\langle v, \bar{x} \rangle \leq 0\} \\ &= \{v \in X^* : F^*(\bar{y} + tv) + F(\bar{x}) - \langle \bar{y} + tv, \bar{x} \rangle \leq 0\} \\ &= \{v \in X^* : \bar{y} + tv \in \partial F(\bar{x})\}. \end{aligned}$$

In other words,

$$A_t(0) = [\partial F(\bar{x}) - \bar{y}]/t.$$

For any $\eta > 0$, one has

$$A_t(0) \subset [\partial_{[\eta t]} F(\bar{x}) - \bar{y}]/t = \eta \cdot \partial_{[\eta t]}^2 F(\bar{x}, \bar{y}),$$

and therefore,

$$w\text{-}\limsup_{t \rightarrow 0^+} A_t(0) \subset w\text{-}\limsup_{t \rightarrow 0^+} \eta \cdot \partial_{[\eta t]}^2 F(\bar{x}, \bar{y}).$$

Since the family $\{\partial_{[\eta t]}^2 F(\bar{x}, \bar{y})\}_{t>0}$ Mosco converges toward $\{g \leq 1\}$, one can write

$$w\text{-}\limsup_{t \rightarrow 0^+} A_t(0) \subset \eta\{g \leq 1\}.$$

Since $\eta > 0$ was arbitrary, one finally gets

$$w\text{-}\limsup_{t \rightarrow 0^+} A_t(0) \subset \bigcap_{\eta>0} \eta\{g \leq 1\} = \{g \leq 0\}.$$

Thus, by applying Lemma 2.1, one ensures the existence of the Mosco limit $(F^*)''(\bar{y}, \bar{x}; \cdot)$. The existence of $F''(\bar{x}, \bar{y}; \cdot)$ follows then by duality arguments as in Do [Do2, Theorem 2.5].

Finally, we need to check that formula (2.23) holds. To see this, observe that

$$\begin{aligned} C &= \{v \in X^* : (F^*)''(\bar{y}, \bar{x}; v) \leq 1\} \\ &= \{v \in X^* : \sigma[v; \partial^2 F^*(\bar{y}, \bar{x})] \leq 1\} \\ &= [\partial^2 F^*(\bar{y}, \bar{x})]^0, \end{aligned}$$

where the notation $\sigma[\cdot; \partial^2 F^*(\bar{y}, \bar{x})]$ refers to the support function of the set $\partial^2 F^*(\bar{y}, \bar{x})$. Similarly, one has

$$\{h \in X : F''(\bar{x}, \bar{y}; h) \leq 1\} = [\partial^2 F(\bar{x}, \bar{y})]^0.$$

It remains now to prove that the set $\{(F^*)''(\bar{y}, \bar{x}; \cdot) \leq 1\}$ is the polar of $\{F''(\bar{x}, \bar{y}; \cdot) \leq 1\}$. But this follows by applying Lemma A in the appendix to the case $s = 2$ and $b = \frac{1}{2}F''(\bar{x}, \bar{y}; \cdot)$. \square

We record next a result whose proof has been implicitly given in the demonstration of the previous theorem.

Proposition 2.3. *Let $F \in \Gamma_0(X)$ be finite at $\bar{x} \in X$, and let $\bar{y} \in \partial F(\bar{x})$. Then, the Mosco-limit $F''(\bar{x}, \bar{y}; \cdot)$ exists if and only if the Mosco limit $(F^*)''(\bar{y}, \bar{x}; \cdot)$ exists. In such a case, the functions $\frac{1}{2}F''(\bar{y}, \bar{x}; \cdot)$ and $\frac{1}{2}(F^*)''(\bar{y}, \bar{x}; \cdot)$ are conjugate to each other (cf. [Do2, Theorem 2.5]), and the sets $\partial^2 F(\bar{x}, \bar{y})$ and $\partial^2 F^*(\bar{y}, \bar{x})$ are polar to each other (cf. [Se3, Lemma 4.6]).*

There are several ways of characterizing the Mosco convergence of a family $\{C_t\}_{t>0}$ of sets in X^* . One can use, for instance, the support function

$$h \in X \mapsto \sigma[h; C_t] := \sup\{\langle v, h \rangle : v \in C_t\}$$

associated to each C_t . For the particular case $C_t = \partial_{[t]}^2 F(\bar{x}, \bar{y})$, it is clear that

$$\sigma[h; \partial_{[t]}^2 F(\bar{x}, \bar{y})] = \frac{F'_{[t]}(\bar{x}; h) - \langle \bar{y}, h \rangle}{t} \quad \text{for all } h \in X,$$

where $F'_{[t]}(\bar{x}; \cdot) := \sigma[\cdot; \partial_{[t]} F(\bar{x})]$ stands for the *approximate directional derivative* of F at \bar{x} .

Corollary 2.1. *Let $F \in \Gamma_0(X)$ be finite at $\bar{x} \in X$, and let $\bar{y} \in \partial F(\bar{x})$. Then, the Mosco limit $F''(\bar{x}, \bar{y}; \cdot)$ exists if and only if the Mosco limit*

$$(2.25) \quad \Delta F(\bar{x}, \bar{y}; \cdot) = \text{M-} \lim_{t \rightarrow 0^+} \frac{F'_{[t]}(\bar{x}; \cdot) - \langle \bar{y}, \cdot \rangle}{t}$$

exists. In such a case one has:

$$(2.26) \quad F''(\bar{x}, \bar{y}; h) = [\Delta F(\bar{x}, \bar{y}; h)]^2 \quad \text{for all } h \in X.$$

Proof. Just combine Theorem 2.1 and [Mo2, Theorem 3.1]. \square

A second way of characterizing the Mosco convergence of a family $\{C_t\}_{t>0}$ is by using the distance function

$$z \in X^* \mapsto d(z; C_t) := \inf\{\|z - v\|_* : v \in C_t\}$$

associated to each $C_t \subset X^*$. Here $\|\cdot\|_*$ stands for the dual norm in X^* associated to the norm $\|\cdot\|$ in X . Recall that a normed space $(X, \|\cdot\|)$ is *Kadec* if for all $\{x_n\} \subset X$ such that $\|x_n\| \rightarrow \|x\|$ and $x_n \xrightarrow{w} x$, one has $x_n \xrightarrow{s} x$.

Corollary 2.2. *Suppose the reflexive Banach space $(X, \|\cdot\|)$ is Kadec. Let $F \in \Gamma_0(X)$ be finite at $\bar{x} \in X$, and let $\bar{y} \in \partial F(\bar{x})$. Then, the following statements are equivalent:*

$$(2.27) \quad \text{the Mosco limit } F''(\bar{x}, \bar{y}; \cdot) \text{ exists;}$$

$$(2.28) \quad \begin{aligned} &\text{there is a nonempty closed convex set } C \subset X^* \text{ such that} \\ &\text{for all } z \in X^*, \lim_{t \rightarrow 0^+} t^{-1} d(\bar{y} + tz; \partial_{[t]} F(\bar{x})) = d(z; C). \end{aligned}$$

When these equivalent conditions hold, one has

$$(2.29) \quad C = \partial^2 F(\bar{x}, \bar{y}) = \{z \in X^* : \lim_{t \rightarrow 0^+} t^{-1} d(\bar{y} + tz; \partial_{[t]} F(\bar{x})) = 0\}.$$

Proof. Condition (2.27) is equivalent to the Mosco convergence of the family $\{\partial_{[t]}^2 F(\bar{x}, \bar{y})\}_{t>0}$. According to [BF, Theorem 3.4], the later condition amounts to the existence of a nonempty closed convex set $C \subset X^*$ such that

$$(2.30) \quad \lim_{t \rightarrow 0^+} d(z; \partial_{[t]}^2 F(\bar{x}, \bar{y})) = d(z; C) \quad \text{for all } z \in X^*.$$

But, a straightforward calculus shows that

$$\begin{aligned} d(z; \partial_{[t]}^2 F(\bar{x}, \bar{y})) &= \inf\{\|z - t^{-1}(v - \bar{y})\|_* : v \in \partial_{[t]} F(\bar{x})\} \\ &= t^{-1} \inf\{\|\bar{y} + tz - v\|_* : v \in \partial_{[t]} F(\bar{x})\} \\ &= t^{-1} d(\bar{y} + tz; \partial_{[t]} F(\bar{x})). \end{aligned}$$

The equivalence between (2.27) and (2.28) is proven in this way. Of course, the set C corresponds to the Mosco limit of the family $\{\partial_{[t]}^2 F(\bar{x}, \bar{y})\}_{t>0}$. Thus, (2.29) follows from Theorem 2.1 and the expression for $d(z; C)$ given in (2.30). \square

A third way of characterizing the Mosco convergence of a family $\{C_t\}_{t>0}$ is based on the projection mapping

$$z \in X^* \mapsto P(z; C_t) = \text{the point in } C_t \text{ which is nearest to } z.$$

Such a characterization applies only when the space X has some additional structure.

Corollary 2.3. *Let X be a reflexive Banach space such that both X and X^* are strictly convex and Kadec. Suppose $F \in \Gamma_0(X)$ is finite at $\bar{x} \in X$, and $\bar{y} \in \partial F(\bar{x})$. Then, the following statements are equivalent:*

$$(2.31) \quad \text{the Mosco limit } F''(\bar{x}, \bar{y}; \cdot) \text{ exists;}$$

$$(2.32) \quad \begin{aligned} &\text{there is a nonempty closed convex set } C \subset X^* \text{ such that} \\ &\text{for all } z \in X^*, \lim_{t \rightarrow 0^+} t^{-1}[P(\bar{y} + tz; \partial_{[t]} F(\bar{x})) - \bar{y}] = P(z; C). \end{aligned}$$

When these equivalent conditions hold, one has

$$(2.33) \quad C = \partial^2 F(\bar{x}, \bar{y}) = \{z \in X^* : \lim_{t \rightarrow 0^+} t^{-1}[P(\bar{y} + tz; \partial_{[t]} F(\bar{x})) - \bar{y}] = z\}.$$

Proof. According to [S, Theorem 2.3], the Mosco convergence of the family $\{\partial_{[t]}^2 F(\bar{x}, \bar{y})\}_{t>0}$ amounts to the existence of a nonempty closed convex set $C \subset X^*$ such that

$$(2.34) \quad \lim_{t \rightarrow 0^+} P(z; \partial_{[t]}^2 F(\bar{x}, \bar{y})) = P(z; C) \quad \text{for all } z \in X^*.$$

But, as a matter of calculus, one has

$$P(z; \partial_{[t]}^2 F(\bar{x}, \bar{y})) = t^{-1}[P(\bar{y} + tz; \partial_{[t]} F(\bar{x})) - \bar{y}].$$

This proves the equivalence between (2.31) and (2.32). As in the proof of Corollary 2.2, the set C is the Mosco limit of the family $\{\partial_{[t]}^2 F(\bar{x}, \bar{y})\}_{t>0}$. Thus, (2.33) follows from Theorem 2.1 and the expression of $P(z; C)$ given in (2.34). \square

Now we come back to the general setting of a reflexive Banach space X , and we look at the set $\partial^2 F(\bar{x}, \bar{y})$ from a different point of view. The purpose of the next theorem is to show that under suitable assumptions, the set $\partial^2 F(\bar{x}, \bar{y})$ can be expressed as the Minkowski sum of a cone $T[\bar{y}; \partial F(\bar{x})]$ and a “second-order subdifferential” $\partial^2 F(\bar{x})$ which does not depend on \bar{y} . Such type of representation formula for $\partial^2 F(\bar{x}, \bar{y})$ has been established in [Se1, Proposition B.1.17] for the case in which the convergence of second-order differential quotients is understood in the pointwise sense. To continue with our discussion we need to introduce some

notations. In what follows we assume for convenience that $F \in \Gamma_0(X)$ is continuous at \bar{x} . The notation

$$N[\bar{y}; \partial F(\bar{x})] = \{h \in X : \langle y - \bar{y}, h \rangle \leq 0 \text{ for all } y \in \partial F(\bar{x})\}$$

refers to the normal cone to $\partial F(\bar{x})$ at \bar{y} , and

$$T[\bar{y}; \partial F(\bar{x})] = \{z \in X^* : \langle z, h \rangle \leq 0 \text{ for all } h \in N[\bar{y}; \partial F(\bar{x})]\}$$

is the tangent cone to $\partial F(\bar{x})$ at \bar{y} .

A natural way of defining a second-order directional derivative $D^2F(\bar{x}; \cdot)$ is by introducing the De la Valle-Poussin second-order differential quotient

$$\delta_t^2 F(\bar{x}; h) = \frac{2}{t} \left[\frac{F(\bar{x} + th) - F(\bar{x})}{t} - DF(\bar{x}; h) \right],$$

where

$$DF(\bar{x}; h) = \lim_{t \rightarrow 0^+} \frac{F(\bar{x} + th) - F(\bar{x})}{t}$$

is the usual first-order directional derivative. The pointwise limit

$$D^2F(\bar{x}; \cdot) = \lim_{t \rightarrow 0^+} \delta_t^2 F(\bar{x}; \cdot),$$

if it exists, is a nonnegative function satisfying

$$D^2F(\bar{x}; 0) = 0 \text{ and } D^2F(\bar{x}; \alpha h) = \alpha^2 D^2F(\bar{x}; h) \text{ for all } \alpha > 0 \text{ and } h \in X.$$

Even if the function $D^2F(\bar{x}; \cdot)$ is not convex in general, one can always introduce the set

$$(2.35) \quad \partial^2 F(\bar{x}) = \{z \in X^* : \langle z, h \rangle \leq [D^2F(\bar{x}; h)]^{1/2} \text{ for all } h \in X\}.$$

Proposition 2.4. *Let $F \in \Gamma_0(X)$ be continuous at $\bar{x} \in X$. Suppose that F is second-order regular at \bar{x} relative to $\bar{y} \in \partial F(\bar{x})$ in the sense that*

$$(2.36) \quad \liminf_{\substack{t \rightarrow 0^+ \\ h' \xrightarrow{h}}} \delta_t^2 F(\bar{x}, \bar{y}; h') = \lim_{t \rightarrow 0^+} \delta_t^2 F(\bar{x}, \bar{y}; h) \quad \text{for all } h \in X.$$

Then, the Mosco limit $F''(\bar{x}, \bar{y}; \cdot)$ exists, and is given by

$$(2.37) \quad F''(\bar{x}, \bar{y}; h) = \begin{cases} D^2F(\bar{x}; h) & \text{if } h \in N[\bar{y}; \partial F(\bar{x})], \\ +\infty & \text{if } h \notin N[\bar{y}; \partial F(\bar{x})]. \end{cases}$$

In particular, the second-order subdifferential $\partial^2 F(\bar{x}, \bar{y})$ admits the inner estimate

$$(2.38) \quad \partial^2 F(\bar{x}, \bar{y}) \supset \partial^2 F(\bar{x}) + T[\bar{y}; \partial F(\bar{x})].$$

If, in addition, $D^2F(\bar{x}; \cdot)$ is convex lower-semicontinuous, then one has the representation formula

$$(2.39) \quad \partial^2 F(\bar{x}, \bar{y}) = cl\{\partial^2 F(\bar{x}) + T[\bar{y}; \partial F(\bar{x})]\},$$

where “cl” denotes the closure operation in X^ .*

Proof. To prove that $\{\delta_t^2 F(\bar{x}, \bar{y}; \cdot)\}_{t>0}$ is Mosco convergent, we take an arbitrary $\{t_n\}_{n \in \mathbb{N}} \rightarrow 0^+$, and we show that $\{\delta_{t_n}^2 F(\bar{x}, \bar{y}; \cdot)\}_{n \in \mathbb{N}}$ Mosco converges toward the function

$$h \in X \mapsto \varphi(h) = \lim_{t \rightarrow 0^+} \delta_t^2 F(\bar{x}, \bar{y}; h).$$

Indeed, for all $h \in X$ and $\{h_n\}_{n \in \mathbb{N}} \xrightarrow{w} h$, one has

$$\varphi(h) = \liminf_{\substack{t \rightarrow 0^+ \\ h' \xrightarrow{w} h}} \delta_t^2 F(\bar{x}, \bar{y}; h') \leq \liminf_{t_n} \delta_{t_n}^2 F(\bar{x}, \bar{y}; h_n).$$

This proves the inequality (2.7) appearing in the definition of the concept of Mosco convergence of functions. To prove the other inequality, that is to say (2.8), it suffices to choose $h_n = h$ for all $n \in \mathbb{N}$. Thus, the Mosco limit $F''(\bar{x}, \bar{y}; \cdot)$ exists, and it is the function φ . Now, a simple calculus shows that

$$\varphi(y) = \begin{cases} D^2 F(\bar{x}; h) & \text{if } DF(\bar{x}; h) = \langle \bar{y}, h \rangle, \\ +\infty & \text{if } DF(\bar{x}; h) \neq \langle \bar{y}, h \rangle. \end{cases}$$

To obtain formula (2.37) it suffices then to observe that

$$h \in N[\bar{y}; \partial F(\bar{x})] \text{ if and only if } DF(\bar{x}; h) = \langle \bar{y}, h \rangle.$$

To evaluate the set

$$\partial^2 F(\bar{x}, \bar{y}) = \partial[F''(\bar{x}, \bar{y}; \cdot)]^{1/2}(0),$$

we write

$$[F''(\bar{x}, \bar{y}; \cdot)]^{1/2} = [D^2 F(\bar{x}; \cdot)]^{1/2} + \psi(\cdot; N[\bar{y}; \partial F(\bar{x})]),$$

where $\psi(\cdot; C)$ denotes the indicator function of $C \subset X^*$. In this way, one obtains the inclusion

$$\begin{aligned} \partial^2 F(\bar{x}, \bar{y}) &\supset \partial[D^2 F(\bar{x}; \cdot)]^{1/2}(0) + \partial\psi(\cdot; N[\bar{y}; \partial F(\bar{x})])(0) \\ &= \partial^2 F(\bar{x}) + T[\bar{y}; \partial F(\bar{x})], \end{aligned}$$

and the representation formula (2.39) if $D^2 F(\bar{x}; \cdot)$ is convex lower-semicontinuous. \square

3. CONVEX INTEGRAL FUNCTIONALS

This section is concerned with the second-order analysis of a convex function I_f having the following structure

$$(3.1) \quad I_f(x) := \int_S f(s, x(s)) \mu(ds) \quad \text{for all } x \in L_E^p.$$

Such a functional I_f is usually referred to as the *integral functional* associated to the integrand f . Some comments concerning the notation used in (3.1) are in order. The set S is assumed to be a nonempty measurable space with σ -algebra \mathcal{A} and finite complete positive measure μ . The letter E denotes a reflexive separable Banach space with topological dual E^* . The canonical pairing between E and E^* is denoted by $\langle \cdot, \cdot \rangle$. In what follows we assume that

$$(3.2) \quad f: S \times E \rightarrow \mathbb{R} \cup \{+\infty\} \text{ is a normal convex integrand.}$$

This means that $f(s, \cdot) \in \Gamma_0(E)$ for all $s \in S$, and f is $\mathcal{A} \otimes \mathcal{B}(E)$ -measurable, where $\mathcal{B}(E)$ is the Borel tribe of E .

As customary, the notation L_E^p (with $p \in [1, +\infty[$) refers to the space of measurable functions $x: S \rightarrow E$ such that $s \in S \mapsto \|x(s)\|_E^p$ is μ -integrable. This space will be paired with $L_{E^*}^q$ by means of the canonical bilinear mapping

$$\langle y, x \rangle := \int_S \langle y(s), x(s) \rangle \mu(ds) \quad \text{for all } y \in L_{E^*}^q, x \in L_E^p.$$

The number $q \in [1, +\infty]$ is of course the conjugate of p , that is to say, $q^{-1} + p^{-1} = 1$. $L_{E^*}^\infty$ corresponds to the space of measurable functions $y: S \rightarrow E^*$ such that $s \in S \mapsto \|y(s)\|_{E^*}$ is μ -essentially bounded (here $\|\cdot\|_{E^*}$ denotes the norm in E^* which is dual to the norm $\|\cdot\|_E$ in E).

To make sure that the functional I_f is proper, we posit

$$(3.3) \quad \begin{aligned} &\forall x \in L_E^p, \text{ there exists a } \mu\text{-integrable function which minorizes} \\ &s \in S \mapsto f(s, x(s)), \end{aligned}$$

$$(3.4) \quad \begin{aligned} &\exists x \in L_E^p \text{ such that } s \in S \mapsto f(s, x(s)) \text{ is majorized} \\ &\text{by some } \mu\text{-integrable function.} \end{aligned}$$

The first-order behaviour of the convex integral functional I_f is well known. Take, for instance, $\bar{x} \in \text{dom } I_f$ and consider the set

$$\partial I_f(\bar{x}) = \{y \in L_{E^*}^q : I_f(x) \geq I_f(\bar{x}) + \langle y, x - \bar{x} \rangle \text{ for all } x \in L_E^p\}.$$

To check whether \bar{y} is a subgradient of I_f at \bar{x} or not, it suffices to apply Rockafellar's rule [Ro3, Theorem 21]:

$$\bar{y} \in \partial I_f(\bar{x}) \Leftrightarrow \bar{y} \in L_{E^*}^q \text{ and } \bar{y}(s) \in \partial f(s, \bar{x}(s)) \text{ for a.e. } s \in S.$$

The notation $\partial f(s, \bar{x}(s))$ stands for the subdifferential at $\bar{x}(s)$ of the function $f(s, \cdot)$ (subdifferentiation will never refer to the integration variable s).

Obtaining second-order information on the behaviour of I_f requires one to impose additional assumptions on the integrand f . To compute the second-order subdifferential $\partial^2 I_f(\bar{x}, \bar{y})$, we start by forming the second-order differential quotient

$$\delta_t^2 I_f(\bar{x}, \bar{y}; h) = \frac{2}{t} \left[\frac{I_f(\bar{x} + th) - I_f(\bar{x})}{t} - \langle \bar{y}, h \rangle \right] \text{ for all } h \in L_E^p.$$

A simple calculus shows that

$$\delta_t^2 I_f(\bar{x}, \bar{y}; h) = I_{\varphi_t}(h) := \int_S \varphi_t(s, h(s)) \mu(ds),$$

where

$$\varphi_t(s, d) := \frac{2}{t} \left[\frac{f(s, \bar{x}(s) + td) - f(s, \bar{x}(s))}{t} - \langle \bar{y}(s), d \rangle \right] \text{ for all } (s, d) \in S \times E.$$

Thus, we need to examine under which conditions on f one can ensure the existence of the Mosco limit

$$I_f''(\bar{x}, \bar{y}; \cdot) = \text{M-} \lim_{t \rightarrow 0^+} I_{\varphi_t}.$$

It is rather simple to check that φ_t is a normal convex integrand (cf. [Do2, Lemma 5.2]). Thus we can invoke the following technical lemma concerning the Mosco convergence of integral functionals associated to normal convex integrands. Mosco-convergence for integral functionals defined over $L_{E^*}^q$ ($1 < q < \infty$) is understood in the sense of Definition 2.1. However, when $q = \infty$, the w -topology in (2.7) must be changed by the w^* -topology $\sigma(L_{E^*}^\infty, L_E^1)$, and the s -topology in (2.8) must be changed by the Mackey topology $\tau(L_{E^*}^\infty, L_E^1)$.

Lemma 3.1 (Salvadori [Sa, Theorem 3.1]). *Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of normal convex integrands $g_n: S \times E \rightarrow \mathbb{R} \cup \{+\infty\}$. Let $g: S \times E \rightarrow \mathbb{R} \cup \{+\infty\}$ be another normal convex integrand. Assume that*

(3.5) *for all $s \in S$, the sequence $\{g_n(s, \cdot)\}_{n \in \mathbb{N}}$ Mosco converges toward $g(s, \cdot)$;*

(3.6) *there exist a sequence $\{h_n\}_{n \in \mathbb{N}}$ in L_E^1 and μ -integrable functions α and β such that $\|h_n(s)\|_E \leq \alpha(s)$, $g_n(s, h_n(s)) \leq \beta(s)$,*

(3.7) *there exists a sequence $\{v_n\}_{n \in \mathbb{N}}$ in $L_{E^*}^\infty$ and bounded measurable functions γ and δ such that $\|v_n(s)\|_{E^*} \leq \gamma(s)$, $g_n^*(s, v_n(s)) \leq \delta(s)$.*

Then, I_{g_n} Mosco converges toward I_g , and $I_{g_n^}$ Mosco converges toward I_{g^*} .*

In the next theorem we give a formula for computing the Mosco limit $I_f''(\bar{x}, \bar{y}; \cdot)$. This formula was established for the first time by Do in his doctoral thesis [Do1], and published later in [Do2, Theorem 5.5]. However, Do assumes that E is a finite dimensional space, and he excludes the case $p = 1$. Under these additional assumptions one can apply a convergence theorem for integral functionals due to Joly and de Thelin [JT, Theorem 1]. Salvadori's result is an extension of [JT, Theorem 1] which allows us to deal with the general case considered in this paper.

Theorem 3.1. *Suppose the integrand f satisfies the basic assumptions (3.2)–(3.4). Let I_f be finite at $\bar{x} \in L_E^p$, and let $\bar{y} \in L_{E^*}^q$ be such that $\bar{y}(s) \in \partial f(s, \bar{x}(s))$ for all $s \in S$. Assume also that for all $s \in S$, the Mosco limit $f''(s, \bar{x}(s), \bar{y}(s); \cdot)$ exists. Then,*

$$(3.8) \quad I_f''(\bar{x}, \bar{y}; h) = \int_S f''(s, \bar{x}(s), \bar{y}(s); h(s)) \mu(ds) \text{ for all } h \in L_E^p,$$

and

$$(3.9) \quad I_{f^*}''(\bar{y}, \bar{x}; v) = \int_S (f^*)''(s, \bar{y}(s), \bar{x}(s); v(s)) \mu(ds) \text{ for all } v \in L_{E^*}^q.$$

Proof. As in [Do2, Theorem 5.5], we observe that $\varphi_t(s, \cdot)$ is a nonnegative function which vanishes at the origin $0 \in E$. In the same way, its Legendre-Fenchel conjugate

$$u \in E^* \mapsto \varphi_t^*(s, u) = \sup_{d \in E} \{\langle u, d \rangle - \varphi_t(s, d)\}$$

is nonnegative and vanishes at $0 \in E^*$. By applying Lemma 3.1 to the family $\{\varphi_t\}_{t>0}$, one concludes that I_{φ_t} Mosco converges toward the integral functional associated to the normal convex integrand $(s, d) \in S \times E \mapsto f''(s, \bar{x}(s), \bar{y}(s); d)$, that is to say,

$$I_f''(\bar{x}, \bar{y}; h) = \left[\text{M-} \lim_{t \rightarrow 0^+} I_{\varphi_t} \right] (h) = \int_S f''(s, \bar{x}(s), \bar{y}(s); h(s)) \mu(ds) \text{ for all } h \in L_E^p.$$

The proof of formula (3.8) is then complete. Now we apply Lemma 3.1 to the family $\{\frac{1}{2}\varphi_t\}_{t>0}$. This yields, in particular, the Mosco convergence of $I_{(\frac{1}{2}\varphi_t)^*}$ toward the integral functional associated to the normal convex integrand $(s, u) \in S \times E \mapsto [\frac{1}{2}f''(s, \bar{x}(s), \bar{y}(s); \cdot)]^*(u)$, i.e.,

$$\left[\text{M-} \lim_{t \rightarrow 0^+} I_{(\frac{1}{2}\varphi_t)^*} \right] (v) = \int_S \left[\frac{1}{2}f''(s, \bar{x}(s), \bar{y}(s); \cdot) \right]^* (v(s)) \mu(ds) \text{ for all } v \in L_{E^*}^q.$$

To obtain formula (3.9) it suffices now to take into account two facts: on the one hand

$$\left[\frac{1}{2}f''(s, \bar{x}(s), \bar{y}(s); \cdot)\right]^*(v(s)) = \frac{1}{2}(f^*)''(s, \bar{y}(s), \bar{x}(s); v(s)),$$

and, on the other hand,

$$\begin{aligned} [I_{(\frac{1}{2}\varphi_t)^*}](v) &= \int_S \left(\frac{1}{2}\varphi_t\right)^*(s, v(s))\mu(ds) \\ &= \int_S \frac{1}{t} \left[\frac{f^*(s, \bar{y}(s) + tv(s)) - f^*(s, \bar{y}(s))}{t} - \langle v(s), \bar{x}(s) \rangle \right] \mu(ds) \\ &= \frac{1}{2}\delta_t^2 I_{f^*}(\bar{y}, \bar{x}; v). \end{aligned}$$

Thus $\frac{1}{2}\delta_t^2 I_{f^*}(\bar{y}, \bar{x}; \cdot)$ Mosco converges toward the function

$$\begin{aligned} v \in L_{E^*}^q &\mapsto \frac{1}{2}I_{f^*}''(\bar{y}, \bar{x}; v) = \left[\text{M-} \lim_{t \rightarrow 0^+} I_{(\frac{1}{2}\varphi_t)^*} \right](v) \\ &= \frac{1}{2} \int_S (f^*)''(s, \bar{y}(s), \bar{x}(s); v(s))\mu(ds). \end{aligned}$$

The proof of formula (3.9) is now complete. \square

Corollary 3.1. *Under the same assumptions as in Theorem 3.1, the functions $\frac{1}{2}I_f''(\bar{x}, \bar{y}; \cdot)$ and $\frac{1}{2}I_{f^*}''(\bar{y}, \bar{x}; \cdot)$ are conjugate to each other.*

Proof. The conjugacy relationship between $\frac{1}{2}I_f''(\bar{x}, \bar{y}; \cdot)$ and $\frac{1}{2}I_{f^*}''(\bar{y}, \bar{x}; \cdot)$ follows from Theorem 3.1 and Rockafellar's result on the Legendre-Fenchel conjugate of a convex integral functional ([Ro2, Theorem 2]). \square

Theorem 3.1 serves also to establish a polarity relationship between the sets

$$\partial^2 I_f(\bar{x}, \bar{y}) := \{z \in L_{E^*}^q : \langle z, h \rangle \leq [I_f''(\bar{x}, \bar{y}; h)]^{1/2} \text{ for all } h \in L_E^p\}$$

and

$$\partial^2 I_{f^*}(\bar{y}, \bar{x}) := \{w \in L_E^p : \langle v, w \rangle \leq [I_{f^*}''(\bar{y}, \bar{x}; v)]^{1/2} \text{ for all } v \in L_{E^*}^q\}.$$

Corollary 3.2. *Under the same assumptions as in Theorem 3.1, the sets $\partial^2 I_f(\bar{x}, \bar{y})$ and $\partial^2 I_{f^*}(\bar{y}, \bar{x})$ are polar to each other.*

Proof. First we write

$$\partial^2 I_f(\bar{x}, \bar{y}) = \{I_f''(\bar{x}, \bar{y}; \cdot) \leq 1\}^0 \quad \text{and} \quad \partial^2 I_{f^*}(\bar{y}, \bar{x}) = \{I_{f^*}''(\bar{y}, \bar{x}; \cdot) \leq 1\}^0.$$

Then, we combine Corollary 3.1 and Lemma A in the appendix. \square

An important contribution of this paper is deriving a formula for the second-order subdifferential $\partial^2 I_f(\bar{x}, \bar{y})$ of the integral functional I_f in terms of the second-order subdifferential $\partial^2 f(s, \bar{x}(s), \bar{y}(s))$ of the integrand f . To write down this formula one needs to introduce first the notation

$$\Lambda := \left\{ \lambda : S \rightarrow \mathbb{R}_+ : \lambda \text{ is measurable and } \int_S [\lambda(s)]^2 \mu(ds) = 1 \right\}.$$

Theorem 3.2. *Under the same assumptions as in Theorem 3.1, one can write* (3.10)

$$\partial^2 I_f(\bar{x}, \bar{y}) = cl \left[\bigcup_{\lambda \in \Lambda} \{z \in L_{E^*}^q : z(s) \in \lambda(s) \partial^2 f(s, \bar{x}(s), \bar{y}(s)) \text{ for a.e. } s \in S\} \right].$$

Proof. According to Proposition 2.1, one has

$$f''(s, \bar{x}(s), \bar{y}(s); d) = \{\sigma[d; A_s]\}^2 \quad \text{for all } d \in E,$$

where

$$d \in E \mapsto \sigma[d; A_s] := \text{Sup}\{\langle u, d \rangle : u \in A_s\}$$

stands for the support function of $A_s := \partial^2 f(s, \bar{x}(s), \bar{y}(s))$. Formula (3.8) amounts to saying that

$$[I_f''(\bar{x}, \bar{y}; h)]^{1/2} = l(h) \quad \text{for all } h \in L_E^p,$$

where

$$l(h) = \left[\int_S \{\sigma[h(s); A_s]\}^2 \mu(ds) \right]^{1/2}.$$

In other words, $\partial^2 I_f(\bar{x}, \bar{y})$ coincides with the support set associated to l , i.e.,

$$\partial^2 I_f(\bar{x}, \bar{y}) = \partial l(0) = \{z \in L_{E^*}^q : \langle z, h \rangle \leq l(h) \text{ for all } h \in L_E^p\}.$$

It remains now to obtain a more explicit expression for the above set. Since A_s contains the origin $0 \in E^*$, the support function $\sigma[\cdot; A_s]$ is nonnegative. Thus, one can write

$$l(h) = \text{Sup}\{l_\lambda(h) : \lambda \in \Lambda\},$$

where

$$l_\lambda(h) := \int_S \lambda(s) \sigma[h(s); A_s] \mu(ds) = \int_S \sigma[h(s); \lambda(s) A_s] \mu(ds).$$

A standard calculus rule on support sets allows us to write

$$(3.11) \quad \partial l(0) = \overline{\text{co}} \left[\bigcup \{\partial l_\lambda(0) : \lambda \in \Lambda\} \right],$$

where

$$\partial l_\lambda(0) = \{z \in L_{E^*}^q : \langle z, h \rangle \leq l_\lambda(h) \text{ for all } h \in L_E^p\},$$

and “ $\overline{\text{co}}$ ” denotes the closed convex hull operator. Since l_λ is a convex integral functional, we apply [Ro3, Theorem 21] to obtain

$$\partial l_\lambda(0) = \{z \in L_{E^*}^q : z(s) \in \lambda(s) A_s \text{ for a.e. } s \in S\}.$$

Now, observe that the set on the right-hand side of (3.11) remains the same if one changes the index set Λ by

$$\Lambda' := \left\{ \lambda : S \rightarrow \mathbb{R}_+ : \lambda \text{ is measurable and } \int_S [\lambda(s)]^2 \mu(ds) \leq 1 \right\}.$$

The later index set is convex, and this fact allows us to show that the operation “ co ” is superfluous in (3.11). This completes the proof of formula (3.10). \square

We end this section by examining Theorem 3.2 under the light of the representation formula (2.39) mentioned in Proposition 2.4. More precisely, we will see that in some particular cases one can decompose $\partial^2 f(s, \bar{x}(s), \bar{y}(s))$ as the Minkowski sum of the cone $T[\bar{y}(s); \partial f(s, \bar{x}(s))]$ and the “second-order subdifferential”

$$\partial^2 f(s, \bar{x}(s)) := \{u \in E^* : \langle u, d \rangle \leq [D^2 f(s, \bar{x}(s); d)]^{1/2} \text{ for all } d \in E\},$$

where $D^2f(s, \bar{x}(s); \cdot)$ stands for the (pointwise) second-order directional derivative of $f(s, \cdot)$ at $\bar{x}(s)$. This decomposition of $\partial^2 f$ leads to a similar one for $\partial^2 I_f$. In fact, $\partial^2 I_f(\bar{x}, \bar{y})$ can be recovered from a component

$$(3.12) \quad Q_f(\bar{x}) := \bigcup_{\lambda \in \Lambda} \{z \in L_{E^*}^q : z(s) \in \lambda(s) \partial^2 f(s, \bar{x}(s)) \text{ for a.e. } s \in S\},$$

which does not depend on \bar{y} , and a conical component

$$(3.13) \quad T_f(\bar{x}, \bar{y}) := \{z \in L_{E^*}^q : z(s) \in T[\bar{y}(s); \partial f(s, \bar{x}(s))]\text{ for a.e. } s \in S\}.$$

Corollary 3.3. *Suppose the integrand f satisfies the basic assumptions (3.2)–(3.4). Let I_f be finite at $\bar{x} \in L_E^p$, and let $\bar{y} \in L_{E^*}^q$ be such that $\bar{y}(s) \in \partial f(s, \bar{x}(s))$ for a.e. $s \in S$. Suppose also that*

$$(3.14) \quad \text{for all } s \in S, f(s, \cdot) \text{ is continuous at } \bar{x}(s);$$

$$(3.15) \quad \text{for all } s \in S, f(s, \cdot) \text{ is second-order regular at } \bar{x}(s) \text{ relative to } \bar{y}(s).$$

Then, one has the inner estimate

$$(3.16) \quad \partial^2 I_f(\bar{x}, \bar{y}) \supset \overline{\text{co}}[Q_f(\bar{x})] + T_f(\bar{x}, \bar{y}).$$

If, in addition, one assumes that

$$(3.17) \quad \text{for all } s \in S, D^2 f(s, \bar{x}(s); \cdot) \text{ is convex lower-semicontinuous,}$$

$$(3.18) \quad \text{for all } h \in L_E^p, \int_S D^2 f(s, \bar{x}(s); h(s)) \mu(ds) < +\infty,$$

then one has the representation formula

$$(3.19) \quad \partial^2 I_f(\bar{x}, \bar{y}) = \text{cl}[Q_f(\bar{x})] + T_f(\bar{x}, \bar{y}).$$

Proof. By combining Proposition 2.4 and Theorem 3.1., one obtains

$$I_f''(\bar{x}, \bar{y}; h) = \int_S D^2 f(s, \bar{x}(s); h(s)) \mu(ds) + \int_S \psi[h(s); N_s] \mu(ds) \text{ for all } h \in L_E^p,$$

where $\psi[\cdot; N_s]$ denote the indicator function of the normal cone

$$N_s := N[\bar{y}(s); \partial f(s, \bar{x}(s))].$$

Thus,

$$(3.20) \quad [I_f''(\bar{x}, \bar{y}; h)]^{1/2} = m(h) + \int_S \sigma[h(s); T_s] \mu(ds),$$

where

$$m(h) = \left[\int_S D^2 f(s, \bar{x}(s); h(s)) \mu(ds) \right]^{1/2},$$

and $\sigma[\cdot; T_s]$ denotes the support function of the tangent cone

$$T_s := T[\bar{y}(s); \partial f(s, \bar{x}(s))].$$

The remaining part of the proof consists in evaluating $\partial[I_f''(\bar{x}, \bar{y}; \cdot)]^{1/2}(0)$, the support set associated to the sublinear function $[I_f''(\bar{x}, \bar{y}; \cdot)]^{1/2}$ given by (3.20). To compute $\partial m(0)$ we proceed as in the proof of Theorem 3.2, that is to say, we write

$$m(h) = \sup_{\lambda \in \Lambda} \int_S \lambda(s) [D^2 f(s, \bar{x}(s); h(s))]^{1/2} \mu(ds).$$

To compute the support set $\partial n(0)$ associated to

$$h \in L_E^p \mapsto n(h) := \int_S \sigma[h(s); T_s] \mu(ds),$$

we apply Rockafellar's formula [Ro3, Theorem 2.1]. One has in general

$$\partial^2 I_f(\bar{x}, \bar{y}) \supset \partial m(0) + \partial n(0),$$

with

$$\partial m(0) \supset \overline{\text{co}} Q_f(\bar{x}) \quad \text{and} \quad \partial n(0) = T_f(\bar{x}, \bar{y}).$$

This proves the inclusion (3.16). Under the hypothesis (3.17) one can write

$$(3.21) \quad \partial^2 I_f(\bar{x}, \bar{y}) = cl[\partial m(0) + \partial n(0)],$$

with

$$(3.22) \quad \partial m(0) = cl Q_f(\bar{x}).$$

If one adds assumption (3.18), then the function $m: L_E^p \rightarrow \mathbb{R}$ is continuous, and $\partial m(0)$ is a compact set in $L_{E^*}^q$. Hence, the Minkowski sum $\partial m(0) + \partial n(0)$ is closed, and formula (3.19) follows from (3.21) and (3.22). \square

Below we illustrate Corollary 3.3 with the help of two elementary examples.

Example 3.1. Consider the convex integral functional

$$x \in L_{\mathbb{R}}^1 \mapsto I_f(x) = \int_0^1 s|x(s)|ds$$

associated to $f(s, \cdot) = s|\cdot|$. If $\bar{x}(s) = 0$ for a.e. $s \in [0, 1]$, then

$$\partial I_f(\bar{x}) = \{y \in L_{\mathbb{R}}^{\infty} : |y(s)| \leq s \text{ for a.e. } s \in [0, 1]\} \quad \text{and} \quad Q_f(\bar{x}) = \{0\}.$$

Let $\bar{y} \in L_{\mathbb{R}}^{\infty}$ be such that $|\bar{y}(s)| = s$ for a.e. $s \in [0, 1]$. Then,

$$\partial^2 f(s, \bar{x}(s), \bar{y}(s)) = \text{sgn}(\bar{y}(s))\mathbb{R}_- = \begin{cases} \mathbb{R}_- & \text{if } \bar{y}(s) = s, \\ \mathbb{R}_+ & \text{if } \bar{y}(s) = -s, \end{cases}$$

and

$$\partial^2 I_f(\bar{x}, \bar{y}) = T_f(\bar{x}, \bar{y}) = \{z \in L_{\mathbb{R}}^{\infty} : z(s) \in \text{sgn}(\bar{y}(s))\mathbb{R}_- \text{ for a.e. } s \in [0, 1]\}.$$

Example 3.2. Consider now $x \in L_{\mathbb{R}}^1 \mapsto I_f(x) = \frac{1}{2} \int_0^1 s[x(s)]^2 ds$. This time $f(s, \cdot) = \frac{s|\cdot|^2}{2}$ is smooth. For \bar{x} as in the previous example, one has $\partial I_f(\bar{x}) = \{\bar{y}\}$, with $\bar{y} \in L_{\mathbb{R}}^{\infty}$ given by $\bar{y}(s) = 0$ for a.e. $s \in [0, 1]$. A simple calculus shows that

$$\partial^2 f(s, \bar{x}(s), \bar{y}(s)) = [-\sqrt{s}, \sqrt{s}], \quad \text{and} \quad T_f(\bar{x}, \bar{y}) = \{0\}.$$

Thus

$$\begin{aligned} \partial^2 I_f(\bar{x}, \bar{y}) &= cl Q_f(\bar{x}) = cl \left[\bigcup_{\lambda \in \Lambda} \{z \in L_{\mathbb{R}}^{\infty} : z(s) \in \lambda(s)[- \sqrt{s}, \sqrt{s}] \text{ for a.e. } s \in [0, 1]\} \right] \\ &= cl \left[\left\{ z \in L_{\mathbb{R}}^{\infty} : \int_0^1 s^{-1} [z(s)]^2 ds \leq 1 \right\} \right]. \end{aligned}$$

4. APPLICATIONS

4.1. Integral functionals depending on a velocity vector. A very common problem in calculus of variations is that of minimizing a cost term of the form

$$C_f(x) := \int_a^b f(s, \dot{x}(s)) ds$$

over some set of admissible trajectories. Usually, the term trajectory is understood as a function in a space like

$$W_E^p := \{x: [a, b] \rightarrow E: x \text{ is absolutely continuous and } \dot{x} \in L_E^p\}.$$

Here $[a, b]$ is interpreted as a time interval, and $x(s) \in E$ represents the state vector at time s . Observe that f does not depend explicitly on x , but on the velocity vector $\dot{x}(s) \in E$. As in Section 3, E denotes an arbitrary reflexive separable Banach space (in most applications; however, E is just a finite dimensional Euclidean space, say $E = \mathbb{R}^N$).

The question addressed in this paragraph is that of quantifying the second-order behaviour of C_f around a given trajectory in W_E^p . To answer this question, we take advantage of the fact that $C_f = I_f \circ A$ is the composition of the integral functional

$$(4.1) \quad u \in L_E^p \mapsto I_f(u) := \int_a^b f(s, u(s)) ds$$

and the linear operator

$$(4.2) \quad \begin{aligned} A: W_E^p &\rightarrow L_E^p \\ x &\mapsto Ax = \dot{x}. \end{aligned}$$

Before passing to the second-order analysis of C_f , we characterize the subdifferential mapping ∂C_f . Recall that W_E^p (with $1 < p < +\infty$) is a reflexive Banach space equipped with the norm

$$\|x\|_{W_E^p} := \|x(a)\|_E + \left\{ \int_a^b \|\dot{x}(s)\|_E^p ds \right\}^{1/p}.$$

In what follows, we identify

$$W_{E^*}^q := \{y: [a, b] \rightarrow E^*: y \text{ is absolutely continuous and } \dot{y} \in L_{E^*}^q\}$$

with the dual space of W_E^p , and use the canonical pairing

$$\langle y, x \rangle := \langle y(a), x(a) \rangle + \int_a^b \langle \dot{y}(s), \dot{x}(s) \rangle ds \quad \text{for all } y \in W_{E^*}^q, x \in W_E^p.$$

Lemma 4.1. *Suppose that the integrand f satisfies the basic assumptions (3.2)–(3.4), with $S = [a, b]$ and $p \in]1, +\infty[$. Let C_f be finite at $x \in W_E^p$. Then, a necessary and sufficient condition for $y \in W_{E^*}^q$ is a subgradient of C_f at x is that*

$$(4.3) \quad y(a) = 0 \text{ and } \dot{y}(s) \in \partial f(s, \dot{x}(s)) \text{ for a.e. } s \in [a, b].$$

Proof. Clearly, the subdifferential of the composite function $I_f \circ A$ admits the inner estimate

$$\partial(I_f \circ A)(x) \supset A^*[\partial I_f(Ax)],$$

where $A^*: L_{E^*}^q \rightarrow W_{E^*}^q$ stands for the adjoint operator associated to the continuous linear operator A . As a matter of calculus, one sees that A^* is given by

$$A^*v = \int_a^{(\cdot)} v(s)ds \quad \text{for all } v \in L_{E^*}^q.$$

Thus,

$$\begin{aligned} A^*[\partial I_f(Ax)] &= \left\{ \int_a^{(\cdot)} v(s)ds : v \in \partial I_f(\dot{x}) \right\} \\ &= \{y \in W_{E^*}^q : y(a) = 0 \text{ and } \dot{y} \in \partial I_f(\dot{x})\}. \end{aligned}$$

This shows that any function $y \in W_{E^*}^q$ satisfying (4.3) is a subgradient of C_f at x . Conversely, take any $y \in \partial C_f(x)$. From the definition of $\partial C_f(x)$ it follows that

$$I_f(\dot{v}) \geq I_f(\dot{x}) + \langle y(a), v(a) - x(a) \rangle + \int_a^b \langle \dot{y}(s), \dot{v}(s) - \dot{x}(s) \rangle ds \quad \text{for all } v \in W_E^p.$$

This yields in particular

$$I_f(u) \geq I_f(\dot{x}) + \int_a^b \langle \dot{y}(s), u(s) - \dot{x}(s) \rangle ds \quad \text{for all } u \in L_E^p,$$

and

$$0 \geq \langle y(a), d - x(a) \rangle \quad \text{for all } d \in E.$$

Hence, $\dot{y} \in \partial I_f(\dot{x})$ and $y(a) = 0$. The proof of the lemma is now complete. \square

Now we are ready to give a formula for the second-order subdifferential $\partial^2 C_f(x, y)$.

Proposition 4.1. *Suppose f satisfies the basic assumptions (3.2)–(3.4), with $S = [a, b]$ and $p \in]1, +\infty[$. Let C_f be finite at $x \in W_E^p$, and let $y \in W_{E^*}^q$ be such that condition (4.3) holds. Moreover, assume that the Mosco limit $f''(s, \dot{x}(s), \dot{y}(s); \cdot)$ exists for all $s \in [a, b]$. Then,*

$$(4.4) \quad C_f''(x, y; h) = \int_a^b f''(s, \dot{x}(s), \dot{y}(s); \dot{h}(s))ds \quad \text{for all } h \in W_E^p,$$

and

$$(4.5) \quad \partial^2 C_f(x, y) = cl \left[\bigcup_{\lambda \in \Lambda} \{z \in W_{E^*}^q : z(a) = 0 \text{ and } \right. \\ \left. \dot{z}(s) \in \lambda(s) \partial^2 f(s, \dot{x}(s), \dot{y}(s)) \text{ for a.e. } s \in [a, b] \} \right].$$

Proof. As a matter of computation one shows that the second-order differential quotient $\delta_t^2 C_f(x, y; \cdot)$ is also a composite function; namely,

$$(4.6) \quad \delta_t^2 C_f(x, y; \cdot) = \delta_t^2 I_f(\dot{x}, \dot{y}; \cdot) \circ A.$$

To prove that the Mosco limit $C_f''(x, y; \cdot)$ exists, we combine Theorem 3.1 and the fact that Mosco convergence is preserved under composition with a surjective continuous linear operator (cf. [Ri, Proposition 3.7]). By passing to the limit in (4.6), one obtains

$$C_f''(x, y; h) = I_f''(\dot{x}, \dot{y}; Ah) \quad \text{for all } h \in W_E^p.$$

Formula (4.4) is proven in this way. Now we compute the second-order subdifferential

$$\partial^2 C_f(x, y) = \partial[C_f''(x, y; \cdot)]^{\frac{1}{2}}(0).$$

By taking into account the equality

$$[C_f''(x, y; \cdot)]^{\frac{1}{2}} = [I_f''(\dot{x}, \dot{y}; \cdot)]^{\frac{1}{2}} \circ A,$$

one gets

$$\partial^2 C_f(x, y) = cl\{A^*[\partial^2 I_f(\dot{x}, \dot{y})]\}.$$

Theorem 3.2 yields in this case

$$(4.7) \quad \partial^2 C_f(x, y) = cl\{A^*[clM]\},$$

with

$$M := \bigcup_{\lambda \in \Lambda} \{w \in L_{E^*}^q : w(s) \in \lambda(s) \partial^2 f(s, \dot{x}(s), \dot{y}(s)) \text{ for a.e. } s \in [a, b]\}.$$

It can be shown that the set on the right-hand side of (4.7) remains unchanged if one drops the closure operation on M , that is to say,

$$\partial^2 C_f(x, y) = cl\{A^*(M)\}.$$

To complete the proof of formula (4.5), observe that

$$\begin{aligned} A^*(M) &= \bigcup_{\lambda \in \Lambda} \{A^*w : w(s) \in \lambda(s) \partial^2 f(s, \dot{x}(s), \dot{y}(s)) \text{ for a.e. } s \in [a, b]\} \\ &= \bigcup_{\lambda \in \Lambda} \{z \in W_{E^*}^q : z(a) = 0 \text{ and } \dot{z}(s) \in \lambda(s) \partial^2 f(s, \dot{x}(s), \dot{y}(s)) \text{ for a.e. } s \in [a, b]\}. \end{aligned}$$

□

Remark 4.1. The condition $y(a) = 0$ in (4.3) reflects the fact that the cost $C_f(x)$ associated to a trajectory x depends only on the velocity vector \dot{x} . That the initial state $x(a)$ is of no relevance in the cost $C_f(x)$, is also reflected at the second-order level. This is why the condition $z(a) = 0$ shows up in formula (4.5).

4.2. Quadratic integral functionals and ellipsoids in L^2 spaces. There is a one-to-one correspondence between the set P_N of symmetric positive semidefinite $N \times N$ matrices, and the set of ellipsoids in \mathbb{R}^N which are bounded and centered at the origin. One can consider, for instance, the correspondence

$$R \in P_N \mapsto E(R) := \partial Q_R^{\frac{1}{2}}(0),$$

where

$$d \in \mathbb{R}^N \mapsto Q_R(d) := \langle d, Rd \rangle$$

is the quadratic form associated to R . The set $E(R)$ is referred to as the ellipsoid associated to R .

In the literature one can find a fairly complete set of calculus rules for combinations of ellipsoids in a finite dimensional setting (cf., for instance, [Se5], [Se6], [Va], [KV]). The purpose of this paragraph is to characterize the ellipsoid associated to

a quadratic integral functional defined on a Hilbert space like $L_N^2 := L_{\mathbb{R}^N}^2$. More precisely, we want to compute the ellipsoid

$$E(\mathfrak{R}) := \partial Q_{\mathfrak{R}}^{\frac{1}{2}}(0) = \{z \in L_N^2 : \langle z, h \rangle \leq [\langle h, \mathfrak{R}h \rangle]^{\frac{1}{2}} \text{ for all } h \in L_N^2\}$$

associated to a self-adjoint positive semidefinite operator $\mathfrak{R}: L_N^2 \rightarrow L_N^2$. This operator \mathfrak{R} is constructed from a family $\{R_s: s \in S\}$ of matrices in P_N as follows:

$$(\mathfrak{R}h)(s) = R_s h(s) \text{ for all } s \in S \text{ and } h \in L_N^2.$$

It is clear to see that \mathfrak{R} is a self-adjoint operator, and that

$$h \in L_N^2 \mapsto Q_{\mathfrak{R}}(h) := \langle h, \mathfrak{R}h \rangle = \int_S \langle h(s), R_s h(s) \rangle \mu(ds)$$

is a nonnegative convex integral functional.

In the next proposition we express $E(\mathfrak{R})$ in terms of the family $\{E(R_s): s \in S\}$.

Proposition 4.2. *Let $\{R_s: s \in S\}$ be a family of matrices in P_N such that*

$$(4.8) \quad \text{for all } d \in \mathbb{R}^N, s \in S \mapsto \langle d, R_s d \rangle \text{ is measurable,}$$

$$(4.9) \quad s \in S \mapsto \|R_s\| := [\text{trace}(R_s^2)]^{1/2} \text{ is bounded.}$$

Then,

$$E(\mathfrak{R}) = cl \left[\bigcup_{\lambda \in \Lambda} \{z \in L_N^2 : z(s) \in \lambda(s) E(R_s) \text{ for a.e. } s \in S\} \right].$$

Proof. Just apply Theorem 3.2 to the integrand $S \times \mathbb{R}^N \mapsto f(s, d) = \frac{1}{2} \langle d, R_s d \rangle$. \square

5. CONCLUSIONS

A generalized second-order derivative $F''(\bar{x}, \bar{y}; \cdot)$ of a function $F \in \Gamma_0(X)$ is obtained by taking the limit (as t goes to 0^+) of a second-order differential quotient like $\delta_t^2 F(\bar{x}, \bar{y}; \cdot)$. There are several reasons why we have chosen the convergence of functions, defined on a reflexive Banach space X , to be understood in the sense of Mosco. Among these reasons we mention:

(a) Mosco convergence yields a second-order derivative concept with good duality and variational properties (cf. [Do2]);

(b) as shown in Theorem 2.1, Mosco convergence allows us to recover the second-order subdifferential $\partial^2 F(\bar{x}, \bar{y})$ from the family $\{\partial_{[t]}^2 F(\bar{x}, \bar{y})\}_{t>0}$ of approximate second-order subdifferentials;

(c) The existence of the Mosco limit $F''(\bar{x}, \bar{y}; \cdot)$ can be characterized in several equivalent ways: in terms of support functions (Corollary 2.1), distance functions (Corollary 2.2), or projection mappings (Corollary 2.3);

(d) Salvadori's lemma [Lemma 3.1] shows that Mosco limits are preserved under the integral sign. This fact allows us to characterize the second-order derivative $I_f''(\bar{x}, \bar{y}; \cdot)$ of a convex integral functional I_f . In Theorem 3.1, we show that Do's formula for $I_f''(\bar{x}, \bar{y}; \cdot)$ remains true even if I_f is defined over a Banach space which is not necessarily reflexive.

Besides the above mentioned results, the present work has further merits. In Theorem 3.2 we establish a formula for the second-order subdifferential $\partial^2 I_f(\bar{x}, \bar{y})$. Then, we apply this formula to a problem of calculus of variations (Proposition 4.1), and to the study of ellipsoidal sets in L^2 -spaces (Proposition 4.2).

We would like to thank the referee for bringing the reference [Le] to our attention. In this reference, A. B. Levy considers a nonconvex integral functional I_f defined over $L^p_{\mathbb{R}^d}$ ($1 \leq p < \infty$). His Theorem 1.4 shows that Do's formula (3.8) remains true if the second derivative $I''_f(\bar{x}, \bar{y}; \cdot)$ is understood in the strong-epigraphical sense, and if f satisfies an extra assumption, namely, a second-order uniform boundedness condition. As mentioned in [Le, Corollary 1.5], this extra assumption is not needed in a convex setting. Levy's results and ours are not really comparable. On the one hand side, Mosco-convergence is harder to obtain than strong epigraphical convergence. On the other hand, Levy goes beyond convexity, but under an extra boundedness assumption. The concept of second-order subdifferential $\partial^2 I_f(\bar{x}, \bar{y})$ makes sense only in a convex setting, and it is not considered in [Le].

APPENDIX

The following abstract lemma has been used in Sections 2 and 3. In a finite dimensional setting, and with a different proof, it appears already in Rockafeller [Ro1, Corollary 15.3.2].

Lemma A. *Let (V, V^*) be a couple of locally convex topological linear spaces in duality by means of a bilinear form $\langle \cdot, \cdot \rangle: V^* \times V \rightarrow \mathbb{R}$. Let $r, s \in]1, +\infty[$ be conjugate numbers, i.e., $r^{-1} + s^{-1} = 1$. Suppose that $b: V \rightarrow [0, +\infty]$ is a convex lower-semicontinuous function satisfying*

$$b(0) = 0 \text{ and } b(\alpha v) = \alpha^s b(v) \text{ for all } \alpha > 0 \text{ and } v \in V.$$

Then, the sets

$$\{sb \leq 1\} := \{v \in V : sb(v) \leq 1\}$$

and

$$\{rb^* \leq 1\} := \{w \in V^* : rb^*(w) \leq 1\}$$

are polar to each other.

Proof. From Fenchel's inequality

$$\langle w, v \rangle \leq b^*(w) + b(v) \text{ for all } w \in V^* \text{ and } v \in V,$$

it follows that

$$\langle w, v \rangle \leq 1 \text{ for all } w \in \{rb^* \leq 1\} \text{ and } v \in \{sb \leq 1\},$$

and consequently,

$$\{rb^* \leq 1\} \subset \{sb \leq 1\}^0.$$

To prove the reverse inclusion, take an arbitrary element w in $\{sb \leq 1\}^0$, that is to say,

$$\langle w, v \rangle \leq 1 \text{ for all } v \in V \text{ such that } sb(v) \leq 1.$$

We need to show that $b^*(w) \leq r^{-1}$. We write then

$$b^*(w) := \sup_{v \in V} \{\langle w, v \rangle - b(v)\} = \sup_{\alpha \geq 0} k_\alpha(w),$$

where

$$k_\alpha(w) := \sup_{v \in V} \{\langle w, v \rangle - b(v) : sb(v) = \alpha^s\}.$$

For $\alpha = 0$, one can show that $k_\alpha(w) = 0$. For $\alpha > 0$, one has

$$\begin{aligned} k_\alpha(w) &= \sup_{v \in V} \{\langle w, v \rangle - \alpha^s/s : sb(v) = \alpha^s\} \\ &= \sup_{v \in V} \{\alpha \langle w, v \rangle - \alpha^s/s : sb(v) = 1\} \\ &\leq \alpha - \alpha^s/s. \end{aligned}$$

In this way we have proven that $k_\alpha(w) \leq r^{-1}$ for all $\alpha \geq 0$, and therefore, $b^*(w) \leq r^{-1}$. \square

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